

Pin⁻(2)-MONOPOLE EQUATIONS AND INTERSECTION FORMS WITH LOCAL COEFFICIENTS OF 4-MANIFOLDS

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ABSTRACT. We introduce a variant of the Seiberg-Witten equations, Pin⁻(2)-monopole equations, and give its applications to intersection forms with local coefficients of 4-manifolds. The first application is an analogue of Froyshov's results on 4-manifolds with definite intersection forms with local coefficients. The second is a local coefficient version of Furuta's 10/8-inequality. As a corollary, we construct nonsmoothable spin 4-manifolds satisfying Rohlin's theorem and the 10/8-inequality.

1. Introduction

K. Froyshov [11] recently proved theorems on intersection forms with local coefficients of 4-manifolds which can be considered as a local coefficient analogue of Donaldson's theorem for definite 4-manifolds [7, 8]. To prove his results, he analyzes the moduli space of SO(3)-instantons, and effectively make use of the existence of a kind of reducibles, *twisted reducibles*, whose stabilizers are $\mathbb{Z}/2$, in order to extract the information on local coefficient cohomology.

The first part of this paper proves an analogue of Froyshov's results by Seiberg-Witten theory. In fact, we prove that, if a closed smooth 4-manifold has a definite intersection form with local coefficient, it should be the standard form.

To state the precise statement, we give some preliminaries. Let X be a closed, connected, oriented smooth 4-manifold. Suppose a double covering \tilde{X} of X is given. Let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$ and $\lambda = \tilde{X} \times_{\{\pm 1\}} \mathbb{R}$ be its associated bundles with fiber \mathbb{Z} and \mathbb{R} . We can consider the cohomology $H^*(X; l)$ with l as bundle of coefficients. Since $l \otimes l = \mathbb{Z}$, we have a homomorphism by the cup product,

$$H^2(X; l) \otimes H^2(X; l) \rightarrow H^4(X; \mathbb{Z}) = \mathbb{Z}.$$

This induces a unimodular quadratic form $Q_{X,l}$ on $H^2(X; l)/\text{torsion}$. Let $b_q(X; l)$ be the l -coefficient q -th Betti number, i.e.,

$$b_q(X; l) = \text{rank } H^q(X; l)/\text{torsion}.$$

The ordinary \mathbb{Z} -coefficient Betti numbers are denoted by $b_q(X)$. The short exact sequence of bundles,

$$0 \rightarrow l \xrightarrow{\cdot 2} l \rightarrow \mathbb{Z}/2 \rightarrow 0,$$

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induces a long exact sequence,

$$\cdots \rightarrow H^q(X; l) \xrightarrow{\cdot 2} H^q(X; l) \rightarrow H^q(X; \mathbb{Z}/2) \rightarrow H^{q+1}(X; l) \rightarrow \cdots.$$

In particular, mod 2 reduction map $H^2(X; l) \rightarrow H^2(X; \mathbb{Z}/2)$ is defined.

Our first theorem is as follows:

Theorem 1.1. *Let X be a closed, connected, oriented smooth 4-manifold. Suppose that a nontrivial \mathbb{Z} -bundle $l \rightarrow X$ satisfies the following:*

- (1) *The intersection form $Q_{X,l}$ is definite.*
- (2) *Let $\lambda = l \otimes \mathbb{R}$. Then $w_1(\lambda)^2$ has a lift in the torsion part of $H^2(X; l)$.*

Then $Q_{X,l}$ is isomorphic to the diagonal form.

The proof of Theorem 1.1 is outlined as follows. For the double covering \tilde{X} associated with l , let $\iota: \tilde{X} \rightarrow \tilde{X}$ be the covering transformation. We consider a Spin^c -structure \tilde{c} on \tilde{X} together with an isomorphism (of order 4) between the pullback Spin^c -structure $\iota^*\tilde{c}$ and the complex conjugation of \tilde{c} . In fact, if we start from a Spin^{c-} -structure on X , a $\text{Pin}^-(2)$ -variant of Spin^c -structure introduced in §3, we obtain an antilinear involution I covering ι on the spinor bundles and the determinant line bundle of \tilde{c} . Then, I acts on the Seiberg-Witten moduli space $\tilde{\mathcal{M}}$ of (\tilde{X}, \tilde{c}) , and we pay attention to its I -invariant part $\tilde{\mathcal{M}}^I$. In fact, on the Spin^{c-} -structure on X , we can define a variant of Seiberg-Witten equations, $\text{Pin}^-(2)$ -monopole equations we call, and we can identify the moduli space of solutions of the $\text{Pin}^-(2)$ -monopole equations, \mathcal{M} , with the I -invariant Seiberg-Witten moduli space $\tilde{\mathcal{M}}^I$. The rest of the argument is analogous to the argument in the alternative proof of Donaldson's theorem by the Seiberg-Witten theory (see e.g. [18, 21]). That is, under the assumptions of Theorem 1.1, we prove the virtual dimension of $\mathcal{M} \cong \tilde{\mathcal{M}}^I$ cannot be greater than $b_1(X; l)$, and obtain an inequality for the characteristic elements of $Q_{X,l}$. Finally, we invoke a theorem of Elkies [9] to prove the form should be the standard form.

In the second part of the paper, the technique of finite dimensional approximation due to Furuta and Bauer [13, 3] is applied to the $\text{Pin}^-(2)$ -monopole map, and we prove a 10/8-type inequality for intersection forms with local coefficients:

Theorem 1.2. *Let X be a closed connected oriented smooth 4-manifold. For any nontrivial \mathbb{Z} -bundle l over X which satisfies $w_1(\lambda)^2 = w_2(X)$, the following inequality holds:*

$$b_+(X; l) \geq -\frac{\text{sign}(X)}{8}.$$

Remark 1.3. (1) In the proof of the 10/8-inequality by Furuta [13], the existence of an extra $\text{Pin}^-(2)$ -action on the Seiberg-Witten theory on the Spin^c -structure associated with a spin structure plays an essential role. Analogously, a key point of the proof of Theorem 1.2 is the existence of an extra gauge symmetry. In fact, there is a larger gauge symmetry on the Spin^{c-} -structure whose associated $\text{O}(2)$ -bundle is $\underline{\mathbb{R}} \oplus \lambda$ (§4(iii)), and such a Spin^{c-} -structure exists if the condition $w_1(\lambda)^2 = w_2(X)$ is satisfied (Proposition 3.4).

(2) Note that $\alpha \cup \alpha = Sq^1(\alpha)$ for $\alpha \in H^1(X; \mathbb{Z}/2)$, and Sq^1 is the Bockstein connecting

homomorphism associated with coefficient sequence

$$(1.4) \quad 0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

([22], 18.12.) For instance, if $w_2(X)$ has an integral lift of order 2, then $w_2(X) = \alpha \cup \alpha$ holds for some α . This follows from comparing the Bockstein sequence associated with (1.4) with another Bockstein sequence associated with the sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

(3) As mentioned above, the proof of Theorem 1.2 use the $\text{Pin}^-(2)$ -monopole map. In fact, the $\text{Pin}^-(2)$ -monopole map can be considered as the I -invariant part of the Seiberg-Witten map of the double covering \tilde{X} . Therefore, we can prove Theorem 1.2 by applying the finite dimensional approximation technique directly to the Seiberg-Witten equations on \tilde{X} with the I -action.

(4) We will give an alternative proof of Theorem 1.1 by using the same technique used in the proof of Theorem 1.2.

As an application of Theorem 1.1 and Theorem 1.2, we construct nonsmoothable 4-manifolds satisfying known constraints on smooth 4-manifolds.

Let us consider the spin cases. For smooth spin 4-manifolds, we know two fundamental theorems, Rohlin's theorem(see e.g.[16]) and Furuta's theorem [13]. Rohlin's theorem tells us that the signature of every closed spin 4-manifold is divisible by 16. On the other hand, Furuta's theorem [13] tells us that every closed smooth spin 4-manifold X with indefinite form satisfies the so-called "10/8-inequality"

$$b_2(X) \geq \frac{5}{4}|\text{sign}(X)| + 2.$$

This inequality is improved by M. Furuta and Y. Kametani [14] in the case when $b_1(X) > 0$. We call the improved inequality in [14] the strong 10/8-inequality.

Theorem 1.5. *There exist nonsmoothable closed spin topological 4-manifolds which have signatures divisible by 16 and satisfy the strong 10/8-inequality.*

The idea of the construction of such nonsmoothable examples is as follows. Let V be any simply-connected topological 4-manifold with even definite form Q_V of rank $16k$, and let X be a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4 's so that the 10/8-inequality is satisfied. Since $b_2(M; l) = 0$ and $w_1(\lambda)^2 = 0$ for a non-trivial \mathbb{Z} -bundle l on $M = T^2 \times S^2$ or T^4 , we can show that X is nonsmoothable by Theorem 1.1. We can also construct similar examples by using Theorem 1.2.

C. Bohr [4] and Lee-Li [17] proved 10/8-type inequalities for non-spin 4-manifolds with even forms. We also construct nonsmoothable non-spin 4-manifolds with even forms satisfying their inequalities.

Theorem 1.6. *There exist nonsmoothable closed non-spin 4-manifolds X with even indefinite forms satisfying $b_2(X) \geq \frac{5}{4}|\text{sign}(X)|$.*

Remark 1.7. One of the results of Bohr [4] and Lee-Li [17] is that the inequality $b_2(X) \geq 5/4|\text{sign}(X)|$ holds for non-spin 4-manifolds X with even indefinite forms whose 2-primary torsion part of $H_1(X; \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2^k$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We construct our examples so that the 2-primary torsion part of $H_1(X; \mathbb{Z})$ is $\mathbb{Z}/2$.

The organization of the paper is as follows. In Section 2, we prove Theorem 1.5 and Theorem 1.6 assuming Theorem 1.1 and Theorem 1.2. In Section 3, we introduce the notion of Spin^c -structures which is a $\text{Pin}^-(2)$ -variant of Spin^c -structures. It is also explained that, if a Spin^c -structure on X is given, then a Spin^c -structure on the double covering \tilde{X} is induced, and the covering transformation of \tilde{X} is covered by antilinear involutions I on the spinor bundles and the determinant line bundle. In Section 4, we introduce $\text{Pin}^-(2)$ -monopole equations, and show that the moduli space of solutions of $\text{Pin}^-(2)$ -monopole equations can be identified with the I -invariant Seiberg-Witten moduli space on the double covering \tilde{X} . We also analyze the structure of $\text{Pin}^-(2)$ -monopole moduli spaces when $b_+(X; \lambda) = 0$. In Section 5, we prove Theorem 1.1. In Section 6, the Bauer-Furuta theory [13, 3] of $\text{Pin}^-(2)$ -monopole map is studied, and Theorem 1.2 is proved by using the equivariant K -theory as in [13, 5]. We also give an alternative proof of Theorem 1.1 by the same technique.

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2. Applications

In this section, we prove Theorem 1.5 and Theorem 1.6 assuming Theorem 1.1 and Theorem 1.2. First, we prove the following. (Cf. [11], Corollary 1.1.)

Theorem 2.1. *Let V be any closed oriented topological 4-manifold which satisfies either of the following:*

- (1) *the intersection form Q_V on $H^2(V; \mathbb{Z})$ is non-standard definite, or*
- (2) *there exists an element $\alpha \in H^1(V; \mathbb{Z}/2)$ so that $\alpha \cup \alpha = w_2(V)$, and the intersection form Q_{V, l_α} satisfies $b_+(V; l_\alpha) < -\text{sign}(V)/8$, where l_α is the \mathbb{Z} -bundle corresponding to α . (If $w_2(V) = 0$, then α may be 0.)*

Let M be a closed oriented 4-manifold which admits a nontrivial \mathbb{Z} -bundle $l' \rightarrow M$ such that $b_2(M; l') = 0$ and $w_1(l')^2 = 0$, where $l' = l' \otimes \mathbb{R}$. Then the connected sum $X = V \# M$ does not admit any smooth structure.

Before proving Theorem 2.1, we will discuss how to construct V and M as in the theorem. One can construct simply-connected examples of V satisfying (1) by Freedman's theory [10]. Examples of V satisfying (2) can be constructed as follows. Let $|E_8|$ be the simply-connected topological 4-manifold whose form is $-E_8$. (This can be also constructed by

Freedman's theory.) Then $V = m|E_8|\#n(S^2 \times S^2)$ with $m > n$ are spin manifolds satisfying (2) with $\alpha = 0$.

As shown in Hambleton-Kreck's paper [15](Proof of Theorem 3), there exist non-spin topological rational homology 4-spheres Σ_0 and Σ_1 with $\pi_1 = \mathbb{Z}/2$ and Kirby-Siebenmann obstructions $\text{ks}(\Sigma_0) = 0$ and $\text{ks}(\Sigma_1) \neq 0$. For instance, an Enriques surface is topologically decomposed into $|E_8|\#(S^2 \times S^2)\#\Sigma_1$. Then $V = m|E_8|\#n(S^2 \times S^2)\#\Sigma_i$ with $m > n + 1$ are non-spin manifolds satisfying (2) with non-zero class $\alpha \in H^1(V; \mathbb{Z}/2) \cong H^1(\Sigma_i; \mathbb{Z}/2) \cong \mathbb{Z}/2$ as follows. First, note that $b_+(V; l_\alpha) = b_+(V) + 1$ in this case. This follows from the following fact: for any \mathbb{Z} -bundle l over a manifold X , let \tilde{X} be the double covering corresponding to l , and let $\underline{\lambda} = l \otimes \mathbb{R}$ considered as a bundle with discrete fibers. Then, we have in general,

$$\begin{aligned} b_0(X) - b_1(X) + b_+(X) &= b_0(X; l) - b_1(X; l) + b_+(X; l), \\ H^*(\tilde{X}; \mathbb{R}) &= H^*(X; \mathbb{R}) \oplus H^*(X; \underline{\lambda}). \end{aligned}$$

Second, since $H^2(\Sigma_i; \mathbb{Z}) = \mathbb{Z}/2$, the integral lift of $w_2(\Sigma_i)$ has order 2. By Remark 1.3(2), the generator $\alpha \in H^1(\Sigma_i; \mathbb{Z}/2) = \mathbb{Z}/2$ should satisfy $w_2(\Sigma_i) = \alpha \cup \alpha$. Note also that $\text{ks}(V) = 0$ if and only if $m + i \equiv 0 \pmod{2}$.

As examples of M , we can take $M = T^2 \times S^2$ or T^4 or their arbitrary connected sum. In fact, $b_2(M; l') = 0$ and $w_1(\lambda')^2 = 0$ for any nontrivial \mathbb{Z} -bundle l' over $M = T^2 \times S^2$ or T^4 . When M is a connected sum of several $T^2 \times S^2$ or T^4 , take l' which is nontrivial on each $T^2 \times S^2$ or T^4 summand.

Proof of Theorem 2.1. Suppose V satisfies (1) and X is smoothable. Take l' as in the assumption, and let $l \rightarrow V\#M$ be the connected sum of a trivial \mathbb{Z} -bundle on V and l' . Then, $H^2(X; l) = H^2(V; \mathbb{Z}) \oplus H^2(M; l')$ and $Q_{X, l} = Q_V$. Note that $w_1(\lambda)^2 = w_1(\lambda')^2 = 0$. By Theorem 1.1, $Q_{X, l}$ should be standard. This is a contradiction. If V satisfies (2), then consider $l = l_\alpha\#l'$ and use Theorem 1.2. \square

Proof of Theorem 1.5. Let V be any simply-connected 4-manifold with even form Q_V of rank $16k$ which satisfies either of the following:

- (1) Q_V is definite, or
- (2) $Q_V \cong m(-E_8) \oplus nH$ and $m > n$, where H is the hyperbolic form.

Then, take a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4 's so that the 10/8-inequality is satisfied. By Theorem 2.1, it is nonsmoothable. \square

Proof of Theorem 1.6. Let $V = m|E_8|\#n(S^2 \times S^2)\#\Sigma_i$ with $m > n + 1$, and take a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4 's. \square

3. Spin^{c-}-structures

In this section, we introduce a variant of Spin^c-structure, Spin^{c-}-structure we call. The notion of Spin^{c-}-structure was introduced to the author by M. Furuta, and a large part of this section is due to him.

3(i). **Spin^{c-} -groups.** Let $\text{Pin}^-(2)$ be the subgroup of $\text{Sp}(1)$ generated by $U(1)$ and j , that is, $\text{Pin}^-(2) = U(1) \cup jU(1)$. There is a two-to-one homomorphism $\varphi_0: \text{Pin}^-(2) \rightarrow O(2)$, which sends $z \in U(1)$ in $\text{Pin}^-(2)$ to $z^2 \in U(1) \subset O(2)$, and j to the reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us define $\text{Spin}^{c-}(n) = \text{Spin}(n) \times_{\{\pm 1\}} \text{Pin}^-(2)$. There is an exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}^{c-}(n) \rightarrow \text{SO}(n) \times O(2) \rightarrow 1.$$

3(ii). **Spin^{c-} -structures.** Let X be a n -dimensional oriented smooth manifold. Fix a Riemannian metric on X , and let $F(X)$ be its $\text{SO}(n)$ -frame bundle. Suppose an $O(2)$ -bundle E over X is given.

Definition 3.1. A Spin^{c-} -structure on (X, E) is a lift of the principal $\text{SO}(n) \times O(2)$ -bundle $F(X) \times_X E$ to a principal $\text{Spin}^{c-}(n)$ -bundle. This is given by the data (P, τ) where P is a $\text{Spin}^{c-}(n)$ -bundle and τ is a bundle isomorphism $P/\{\pm 1\} \rightarrow F(X) \times_X E$.

Remark 3.2. More generally, one can define a Spin^{c-} -structure on the pair (V, E) of an $\text{SO}(n)$ -bundle V and an $O(2)$ -bundle E over X as a $\text{Spin}^{c-}(n)$ -lift of $V \times_X E$.

Remark 3.3. Let G_0 be the identity component of $\text{Spin}^{c-}(n)$. Then G_0 is isomorphic to $\text{Spin}^c(n)$, and $\tilde{X} = P/G_0 \rightarrow X$ is a double covering. Note that the determinant line bundle $\det E$ of E is isomorphic to $\tilde{X} \times_{\{\pm 1\}} \mathbb{R}$, where $\{\pm 1\}$ acts on \mathbb{R} by multiplication.

Proposition 3.4. *There exists a Spin^{c-} -structure on $F(X) \times_X E$ if and only if $w_2(TX) = w_2(E) + w_1(E)^2$.*

Proof. Note that the image of $\text{Pin}^-(2) \subset \text{Sp}(1) = \text{Spin}(3)$ by the canonical homomorphism $\text{Spin}(3) \rightarrow \text{SO}(3)$ is a copy of $O(2)$ embedded in $\text{SO}(3)$. This embedding $O(2) \subset \text{SO}(3)$ is given by $A \mapsto A \oplus \det A$. By using this embedding, embed $\text{SO}(n) \times O(2)$ in $\text{SO}(n+3)$. Then we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \text{Spin}^{c-}(n) & \longrightarrow & \text{SO}(n) \times O(2) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \text{Spin}(n+3) & \longrightarrow & \text{SO}(n+3) \longrightarrow 1. \end{array}$$

The diagram leads to a commutative diagram of fibrations

$$\begin{array}{ccccccc} K(\mathbb{Z}_2, 1) & \longrightarrow & B\text{Spin}^{c-}(n) & \longrightarrow & B\text{SO}(n) \times BO(2) & \longrightarrow & K(\mathbb{Z}_2, 2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Z}_2, 1) & \longrightarrow & B\text{Spin}(n+3) & \longrightarrow & B\text{SO}(n+3) & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2). \end{array}$$

From these, we see that

$$w_2(TX \oplus E \oplus \det E) = w_2(X) + w_2(E) + w_1(E)^2 = 0$$

is the required condition. \square

Remark 3.5. Let $l \rightarrow X$ be a \mathbb{Z} -bundle over X , and $\lambda = l \otimes \mathbb{R}$. The isomorphism classes of $O(2)$ -bundles E whose determinant line bundles $\det E$ are isomorphic to λ are classified by their twisted first Chern classes $\tilde{c}_1(E) \in H^2(X; l)$. See [11], Proposition 2.2. Note also that $\tilde{c}_1(E) = 0$ if and only if E is isomorphic to $\underline{\mathbb{R}} \oplus \lambda$, where $\underline{\mathbb{R}}$ is a trivial \mathbb{R} -bundle over X .

We concentrate on the case when $n = 4$ below. Let \mathbb{H}_T be a $\text{Spin}^{c-}(4)$ -module which is a copy of \mathbb{H} as a vector space, such that the action of $[q_+, q_-, u] \in \text{Spin}^{c-}(4) = (\text{Sp}(1) \times \text{Sp}(1)) \times_{\{\pm 1\}} \text{Pin}(2)$ on $v \in \mathbb{H}_T$ is given by $q_+ v q_-^{-1}$. Then, the associated bundle $P \times_{\text{Spin}^{c-}(4)} \mathbb{H}_T$ is identified with the tangent bundle TX .

Similarly, let $\bar{\varphi}: \text{Spin}^{c-}(4) \rightarrow O(2)$ be the homomorphism defined from $\varphi_0: \text{Pin}^-(2) \rightarrow O(2)$. Then the associated bundle $P \times_{\bar{\varphi}} O(2)$ is identified with E .

Let us consider $\text{Spin}^{c-}(4)$ -modules \mathbb{H}_+ and \mathbb{H}_- which are copies of \mathbb{H} as vector spaces, such that the action of $[q_+, q_-, u] \in \text{Spin}^{c-}(4)$ on $\phi \in \mathbb{H}_{\pm}$ is given by $q_{\pm} \phi u^{-1}$. Then, one can obtain the associated bundles $S^{\pm} = P \times_{\text{Spin}^{c-}(4)} \mathbb{H}_{\pm}$. These are *positive* and *negative spinor bundles* for the Spin^{c-} -structure.

The Clifford multiplication $\rho_{\mathbb{R}}: \Omega^1(X) \times \Gamma(S^+) \rightarrow \Gamma(S^-)$ is defined via $\text{Spin}^{c-}(4)$ -equivariant map $\mathbb{H}_T \times \mathbb{H}_+ \rightarrow \mathbb{H}_-$ defined by $(v, \phi) \mapsto \bar{v}\phi$. Later we will need a *twisted complex* version of the Clifford multiplication defined as follows. Let G_0 be the identity component of $\text{Spin}^{c-}(4)$. Then G_0 is isomorphic to $\text{Spin}^c(4)$, and $\text{Spin}^{c-}(4)/G_0 \cong \{\pm 1\}$. Let $\varepsilon: \text{Spin}^{c-}(4) \rightarrow \text{Spin}^{c-}(4)/G_0$ be the projection, and let $\text{Spin}^{c-}(4)/G_0 \cong \{\pm 1\}$ act on \mathbb{C} by complex conjugation. Then $\text{Spin}^{c-}(4)$ acts on \mathbb{C} via ε and complex conjugation. Define

$$\rho_0: \mathbb{H}_T \otimes_{\mathbb{R}} \mathbb{C} \times \mathbb{H}_+ \rightarrow \mathbb{H}_-$$

by $\rho_0(v \otimes a, \phi) = \bar{v}\phi\bar{a}$. This ρ_0 is $\text{Spin}^{c-}(4)$ -equivariant. Let us define the bundle K over X by $K = \tilde{X} \times_{\{\pm 1\}} \mathbb{C}$ where $\{\pm 1\}$ acts on \mathbb{C} by complex conjugation. Then we can define via ρ_0 the Clifford multiplication

$$(3.6) \quad \rho: \Omega^1(X; K) \times \Gamma(S^+) \rightarrow \Gamma(S^-).$$

Note that $K = \underline{\mathbb{R}} \oplus i\lambda$, where $\underline{\mathbb{R}}$ is a trivial \mathbb{R} -bundle. By restricting ρ to $\underline{\mathbb{R}}$, $\rho_{\mathbb{R}}$ is recovered. By restricting ρ to $i\lambda$, we obtain

$$\rho: \Omega^1(X; i\lambda) \times \Gamma(S^+) \rightarrow \Gamma(S^-).$$

3(iii). The relation with Spin^c -structures on the double covering. In this subsection, we write $\text{Spin}^{c-}(4)$ as G . Note that G has two connected components G_0 and G_1 , and the identity component G_0 is $\text{Spin}^c(4)$. If a Spin^{c-} -structure (P, τ) on a 4-manifold X is given, then $\tilde{X} = P/G_0$ gives a double covering $\pi: \tilde{X} \rightarrow X$. Then, we have a G_0 -bundle $P \rightarrow P/G_0 = \tilde{X}$. The pull-back bundle π^*E has an $\text{SO}(2)$ -reduction L , and a bundle isomorphism $\tilde{\tau}: P/\{\pm 1\} \rightarrow F(\tilde{X}) \times_{\tilde{X}} L$ is induced from τ , where $F(\tilde{X}) = \pi^*F(X)$, which can be considered as the frame bundle over \tilde{X} for the pull-back metric. The G_0 -bundle P over \tilde{X} and $\tilde{\tau}$ define an ordinary Spin^c -structure \tilde{c} on \tilde{X} .

Let $\iota: \tilde{X} \rightarrow \tilde{X}$ be the covering transformation, and define J by

$$J = [1, j^{-1}] \in G_1 = \text{Spin}(4) \times_{\{\pm 1\}} j \text{U}(1).$$

Then the right J -action on $P \rightarrow \tilde{X}$ covers the ι -action. Although the J -action is not a G_0 -bundle automorphism of $P \rightarrow \tilde{X}$, it can be considered as the composition of the following two maps of G_0 -bundles:

- A G_0 -bundle map covering the ι -action, $\tilde{\iota}: P \rightarrow \bar{P}$, where \bar{P} is the G_0 -bundle for the complex conjugate Spin^c -structure of \tilde{c} .
- The complex conjugation, $\alpha: \bar{P} \rightarrow P$, covering the identity map of \tilde{X} .

To see this, let us consider the pull-back G -bundle $\pi^*P \rightarrow \tilde{X}$. Then

$$\pi^*P = P \times_G (G/G_0 \times G) = P \times_G (\{\pm 1\} \times G) = P \times_{G_0} G.$$

The bundle $P \times_{G_0} G$ has two components: $P \times_{G_0} G = P_0 \sqcup P_1$, where $P_i = P \times_{G_0} G_i$ for $i = 0, 1$. Since the right G_0 -action on π^*P preserves P_0 and P_1 , P_0 and P_1 are considered as G_0 -bundles over \tilde{X} by this G_0 -action. Note that $P_0 = P \times_{G_0} G_0$ is isomorphic to P as G_0 -bundles over \tilde{X} . The isomorphism $\alpha_0: P \times_{G_0} G_0 \rightarrow P$ is given by $\alpha_0([p, g]) = pg$ for $[p, g] \in P \times_{G_0} G_0$.

On the other hand, P_1 can be identified with the complex conjugation \bar{P} of P as follows: For $g = [s, u] \in \text{Spin}(4) \times_{\{\pm 1\}} \text{U}(1) = G_0$, let \bar{g} be $[s, u^{-1}]$. Every element $g' \in \text{Spin}(4) \times_{\{\pm 1\}} j \text{U}(1) = G_1$ can be written as $g' = J^{-1}g = \bar{g}J^{-1}$ for some $g \in G_0$. Let us define the fiber-preserving diffeomorphism $\alpha_1: P_1 = P \times_{G_0} G_1 \rightarrow P$ by $\alpha_1([p, \bar{g}J^{-1}]) = p\bar{g}$. Then, for $q \in P_1$ and $h \in G_0$,

$$\alpha_1(qh) = \alpha_1(q)\bar{h}.$$

This means $P_1 \cong \bar{P}$ as G_0 -bundles. Let us define $\alpha: P_1 \rightarrow P_0$ by $\alpha = \alpha_0^{-1} \circ \alpha_1$. Explicitly, $\alpha([p, g']) = [p, g'J]$.

The map $\iota: \tilde{X} \rightarrow \tilde{X}$ has a natural lift $\tilde{\iota}: P \times_{G_0} G \rightarrow P \times_{G_0} G$ given by $\tilde{\iota}([p, g]) = [pJ, J^{-1}g]$. Note that $\tilde{\iota}$ exchanges the components P_0 and P_1 . Then the J -action on P can be identified with the composition $\alpha \circ \tilde{\iota}: P_0 \rightarrow P_0$.

The J -action also induces antilinear automorphisms, denoted by I , on the spinor bundles $\tilde{S}^\pm = P \times_{G_0} \mathbb{H}_\pm$ given by $I([p, \phi]) = [pJ, J^{-1} \cdot \phi] = [pJ, \phi j^{-1}]$. (In the expression $J^{-1} \cdot \phi$, " \cdot " means the G -action on \mathbb{H}^\pm .) Since $J^2 \in G_0$, $I^2([p, \phi]) = [pJ^2, J^{-2} \cdot \phi] = [p, \phi]$. Therefore I is an antilinear involution on each of spinor bundles. The relation between the Spin^c -spinor bundles \tilde{S}^\pm over \tilde{X} and the Spin^{c-} -spinor bundles S^\pm over X is given by

$$(3.7) \quad \tilde{S}^\pm \cong \pi^*S^\pm, \quad S^\pm \cong \tilde{S}^\pm/I.$$

Similarly, the J -action induces an antilinear involution of the determinant line bundle, also denoted by I . This can be seen from the construction above, or noticing the following. Note that $\lambda = \tilde{X} \times_{\{\pm 1\}} \mathbb{R} \rightarrow X$ is isomorphic to the determinant \mathbb{R} -bundle of E . Let $E_0 \rightarrow X$ be the \mathbb{R}^2 -bundle associated to E . Then, the determinant \mathbb{C} -bundle L_0 of \tilde{c} can be identified with the pull-back π^*E_0 as *real* vector bundles, and the involution ι lifts to $L_0 \cong \pi^*E_0$ as an involutive antilinear bundle automorphism.

Remark 3.8. By using $\text{Pin}^+(2)$ (instead of $\text{Pin}^-(2)$), we can define analogous objects, Spin^{c+} -structures. A definition of $\text{Pin}^+(2)$ is given as follows: Let us consider the embedding

of O(2) into SO(5) defined by

$$O(2) \ni A \mapsto A \oplus \det A \oplus \det A \oplus \det A \in SO(5),$$

and let $\varphi: \text{Spin}(5) \rightarrow \text{SO}(5)$ be the canonical homomorphism. Then $\text{Pin}^+(2)$ is defined by $\text{Pin}^+(2) = \varphi^{-1}(O(2))$. It can be seen that $\text{Pin}^+(2)$ is isomorphic to $O(2)$ which is considered as a double covering of $O(2)$. (On the other hand, $\text{Pin}^-(2)$ can be defined via the embedding of $O(2)$ into $\text{SO}(3)$ defined by $O(2) \ni A \mapsto A \oplus \det A \in \text{SO}(3)$.)

In the case of Spin^{c+} -structures also, one can construct a Spin^c -structure \tilde{c} associated to it on a double covering \tilde{X} of X . But the covering transformation ι lifts on the spinor bundles as a $\mathbb{Z}/4$ -action.

4. Pin⁻(2)-monopole equations

In this section, we introduce Pin⁻(2)-monopole equations, and develop the Pin⁻(2)-monopole gauge theory. The whole story is almost parallel to the ordinary Seiberg-Witten case.

4(i). **Dirac operators.** Let X be a closed connected oriented smooth 4-manifold, E be a $O(2)$ -bundle over X , and $\lambda = \det E$. We suppose λ is a nontrivial bundle throughout the rest of the paper. Fix a Riemannian metric on X . Suppose a Spin^{c-} -structure (P, τ) on (X, E) is given. If an $O(2)$ -connection A on E is given, then A and the Levi-Civita connection induces a $\text{Spin}^{c-}(4)$ -connection on P , and we can define the Dirac operator via the Clifford multiplication ρ of (3.6) as

$$D_A: \Gamma(S^+) \rightarrow \Gamma(S^-).$$

The Dirac operator D_A also have properties similar to the ordinary Dirac operators. If A' is another $O(2)$ -connection on E , then $a = A - A'$ is in $\Omega^1(X; i\lambda)$, and the relation of Dirac operators of A and $A' = A + a$ is given via ρ by

$$D_{A+a}\Phi = D_A\Phi + \frac{1}{2}\rho(a)\Phi.$$

While the ordinary spinor bundles are equipped with the canonical hermitian inner products, the spinor bundles for a Spin^{c-} -structure do not have such hermitian inner products. However, the pointwise *twisted* hermitian inner product

$$(4.1) \quad \langle \cdot, \cdot \rangle_{K,x}: S_x^\pm \times S_x^\pm \rightarrow K_x$$

is naturally defined, where the objects with the subscription x means the fibers over $x \in X$, and $K = \tilde{X} \times_{\{\pm 1\}} \mathbb{C}$. The precise meaning is as follows: Let \tilde{S}^\pm be the spinor bundles of the associated Spin^c -structure on the double covering \tilde{X} . Then the canonical hermitian inner product of \tilde{S}^\pm can be given as the bundle homomorphisms

$$(4.2) \quad \tilde{S}^\pm \otimes \tilde{S}^\pm \rightarrow \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}}$ is a trivial bundle $\tilde{X} \times \mathbb{C}$. The diagonal action of I on $\tilde{S}^\pm \otimes \tilde{S}^\pm$ is an involution, also denoted by I . Let us define the involution I on $\underline{\mathbb{C}} = \tilde{X} \times \mathbb{C}$ by $I(x, v) = (\iota x, \bar{v})$, where

\bar{v} is the complex conjugation of v . Then (4.2) is I -equivariant. Dividing (4.2) by I , we obtain the bundle homomorphism

$$S^\pm \otimes S^\pm \rightarrow K,$$

which gives the twisted hermitian inner product (4.1).

The real part of (4.1)

$$\langle \cdot, \cdot \rangle_{\mathbb{R},x} = \operatorname{Re} \langle \cdot, \cdot \rangle_{K,x}$$

defines a real inner product on S^\pm . Then it is easy to see that the Dirac operator is formally self-adjoint with respect to the L^2 -inner product induced from $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. (Cf. [19], Lemma 3.3.3.)

Proposition 4.3. *Suppose X is closed and a Spin^{c-} -structure on X is given. Then its Dirac operator is formally self-adjoint in the sense that*

$$(\Phi, D_A \Psi)_{L^2} = (D_A \Phi, \Psi)_{L^2},$$

where

$$(\Phi_1, \Phi_2)_{L^2} = \int_X \langle \Phi_1, \Phi_2 \rangle_{\mathbb{R}} d\operatorname{vol}.$$

4(ii). **$\operatorname{Pin}^-(2)$ -monopole equations.** The curvature F_A of A is an element of $\Omega^2(X; i\lambda)$. The space of $i\lambda$ -valued self-dual forms, $\Omega^+(X; i\lambda)$, is also associated to the $\operatorname{Spin}^{c-}(4)$ -bundle P as follows. Let $\varepsilon: \operatorname{Pin}^-(2) \rightarrow \operatorname{Pin}^-(2)/\operatorname{U}(1) \cong \{\pm 1\}$ be the projection, and let $\operatorname{Spin}^{c-}(4)$ act on $\operatorname{im} \mathbb{H}$ by $v \in \operatorname{im} \mathbb{H} \rightarrow \varepsilon(u)q_+ v q_+^{-1}$ for $[q_+, q_-, u] \in \operatorname{Spin}^{c-}(4)$. Then the space of sections of the associated bundle $P \times_{\operatorname{Spin}^{c-}(4)} \operatorname{im} \mathbb{H}$ is isomorphic to $\Omega^+(X; i\lambda)$. For $\phi \in \mathbb{H}_+$, $\phi i \bar{\phi} \in \operatorname{im} \mathbb{H}$, and $\operatorname{Spin}^{c-}(4)$ acts on it similarly. Thus, one can define a quadratic map

$$q: \Gamma(S^+) \rightarrow \Omega^+(X; i\lambda).$$

Let $\mathcal{A}(E)$ be the space of $\operatorname{O}(2)$ -connections on E . Then $\operatorname{Pin}^-(2)$ -monopole equations for $(A, \Phi) \in \mathcal{A}(E) \times \Gamma(S^+)$ are defined by

$$(4.4) \quad \begin{cases} D_A \Phi = 0, \\ F_A^+ = q(\Phi), \end{cases}$$

where F_A^+ is the self-dual part of the curvature F_A .

As in the case of the ordinary Seiberg-Witten equations, it is convenient to work in Sobolev spaces. Fix $k \geq 4$, and take L_k^2 -completion of $\mathcal{A}(E) \times \Gamma(S^+)$. The $\operatorname{Pin}^-(2)$ -monopole equations (4.4) are assumed as equations for L_k^2 -connections/spinors.

4(iii). **Gauge transformations.** The gauge transformation group \mathcal{G} is defined as the space of $\operatorname{Spin}^{c-}(4)$ -equivariant diffeomorphisms of P covering the identity map of the quotient $P/\operatorname{Pin}^-(2)$. Then, \mathcal{G} can be identified with $\Gamma(P \times_{\operatorname{ad}} \operatorname{Pin}^-(2))$, where ad means the adjoint representation on $\operatorname{Pin}^-(2)$ by the $\operatorname{Pin}^-(2)$ -component of $\operatorname{Spin}^{c-}(4)$. Note that $\operatorname{Lie} \mathcal{G} \cong \Gamma(P \times_{\operatorname{ad}} i\mathbb{R}) \cong \Omega^0(X; i\lambda)$. We take L_{k+1}^2 -completion of \mathcal{G} .

Let us look at \mathcal{G} more closely. Recall that $\text{Pin}^-(2) = \text{U}(1) \cup j \text{U}(1)$. For $u, z \in \text{U}(1)$, note that

$$(4.5) \quad \begin{aligned} \text{ad}_z(u) &= zu z^{-1} = u, \\ \text{ad}_{jz}(u) &= jzu z^{-1} j^{-1} = u^{-1}, \\ \text{ad}_z(ju) &= z^2 j u = z^2 u^{-1} j, \\ \text{ad}_{jz}(ju) &= z^{-2} j u^{-1} = z^{-2} u j. \end{aligned}$$

Therefore the adjoint action preserves the component of $\text{Pin}^-(2)$. Then \mathcal{G} can be decomposed into $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$, where $\mathcal{G}_0 = \Gamma(P \times_{\text{ad}} \text{U}(1))$ and $\mathcal{G}_1 = \Gamma(P \times_{\text{ad}} j \text{U}(1))$. To understand \mathcal{G}_0 and \mathcal{G}_1 , we note the next proposition.

Proposition 4.6. *The bundle $P \times_{\text{ad}} \text{U}(1)$ is identified with $\tilde{X} \times_{\{\pm 1\}} \text{U}(1)$, where $\{\pm 1\}$ acts on $\text{U}(1)$ by complex conjugation. The bundle $P \times_{\text{ad}} j \text{U}(1)$ is identified with the bundle $S(E)$ of unit vectors of E .*

Proof. By (4.5), the adjoint action of $G = \text{Spin}^c(4)$ on $\text{U}(1)$ is given by complex conjugation via the projection $G \rightarrow G/G_0 = \{\pm 1\}$. Therefore,

$$P \times_{\text{ad}} \text{U}(1) = P/G_0 \times_{\{\pm 1\}} \text{U}(1) = \tilde{X} \times_{\{\pm 1\}} \text{U}(1).$$

Let G act on \mathbb{C} as follows: For $g = [s, u] \in \text{Spin}(4) \times_{\{\pm 1\}} \text{U}(1)$ and $w \in \mathbb{C}$, define the g -action on w by $g \cdot w = z^2 w$. For $J' = [1, j] \in \text{Spin}(4) \times_{\{\pm 1\}} j \text{U}(1)$ and $w \in \mathbb{C}$, define the J' -action on w by $J' \cdot w = \bar{w}$. Then the associated bundle $P \times_G \mathbb{C}$ is isomorphic to E . Let us embed $j \text{U}(1)$ into \mathbb{C} by

$$j \text{U}(1) \ni ju = u^{-1} j \mapsto u^{-1} \in \text{U}(1) \subset \mathbb{C}.$$

By (4.5), this gives the identification between $P \times_{\text{ad}} j \text{U}(1)$ and $S(E)$. □

In fact, \mathcal{G}_1 is empty except one case.

Proposition 4.7. $\mathcal{G}_1 = \emptyset$ if and only if $\tilde{c}_1(E) \neq 0$.

Proof. By Proposition 4.6, $\mathcal{G}_1 \cong \Gamma(S(E))$, and $\tilde{c}_1(E) = 0$ if and only if E is isomorphic to $\underline{\mathbb{R}} \oplus i\lambda$ as $\text{O}(2)$ -bundles with determinant line bundle λ . (Recall Remark 3.5.) □

The \mathcal{G} -action on $\mathcal{A}(E) \times \Gamma(S^+)$ is given by $g(A, \Phi) = (A - 2g^{-1}dg, g\Phi)$, for $g \in \mathcal{G}$ and $(A, \Phi) \in \mathcal{A}(E) \times \Gamma(S^+)$. If $\Phi \neq 0$, then \mathcal{G} -action on (A, Φ) is free, and such an (A, Φ) is called an *irreducible*. On the other hand, (A, Φ) with $\Phi \equiv 0$ is called a *reducible*. The stabilizer of the \mathcal{G} -action on $(A, 0)$ is the subgroup of constant sections $\{\pm 1\} \subset \mathcal{G}_0$, unless $E \cong \underline{\mathbb{R}} \oplus \lambda$ and A is flat. If $E \cong \underline{\mathbb{R}} \oplus \lambda$ and A is flat, then the stabilizer is generated by the constant section $j \in \mathcal{G}_1$, and is isomorphic to $\mathbb{Z}/4$.

4(iv). **Moduli spaces.** Let us define the moduli spaces \mathcal{M} and \mathcal{M}_0 of $\text{Pin}^-(2)$ -monopoles as follows:

$$\mathcal{M} = \{ \text{solutions to (4.4)} \} / \mathcal{G}, \quad \mathcal{M}_0 = \{ \text{solutions to (4.4)} \} / \mathcal{G}_0.$$

Then, $\mathcal{M}_0 = \mathcal{M}$ unless $\tilde{c}_1(E) = 0$. If $\tilde{c}_1(E) = 0$, then \mathcal{M}_0 is a double covering of \mathcal{M} .

Proposition 4.8. *The moduli spaces \mathcal{M} and \mathcal{M}_0 are compact.*

For the Dirac operators of Spin^c -structures, one can readily prove the Weitzenböck formula (see [19], Proposition 5.1.5),

$$(4.9) \quad D_A^2 \phi = \nabla_A^* \nabla_A \phi + \frac{\kappa}{4} \phi + \frac{\rho(F_A)}{2} \phi,$$

where κ is the scalar curvature of the metric on X . With this understood, the proof of Proposition 4.8 is parallel to the case of the ordinary Seiberg-Witten theory. The compactness of \mathcal{M} can be seen also from the relation with the Seiberg-Witten theory on the double covering as in the next subsection.

4(v). **The relation with the Seiberg-Witten theory on the double covering.** Let $\mathcal{A}(E)$ be the space of $\text{O}(2)$ -connections on E . As explained in §3(iii), for a Spin^c -structure on (X, E) , it is induced a Spin^c -structure \tilde{c} on the double covering \tilde{X} associated to $\lambda = \det E$. Let $\pi: \tilde{X} \rightarrow X$ be the projection and $\iota: \tilde{X} \rightarrow \tilde{X}$ be the covering transformation. Let \tilde{S}^\pm be the spinor bundles of \tilde{c} , L be the determinant line bundle of \tilde{c} , and $\mathcal{A}(L)$ be the space of $\text{U}(1)$ -connections on L . In this situation, the I -action on $\mathcal{C} := \mathcal{A} \times \Gamma(\tilde{S}^+)$ is induced from the I -action on \tilde{S}^\pm and L . Then, by (3.7),

$$\Gamma(S^\pm) \cong \Gamma(\tilde{S}^\pm)^I.$$

The relation of $\mathcal{A}(E)$ and $\mathcal{A}(L)$ is given as follows. An $\text{O}(2)$ -connection A on E and the Levi-Civita connection determine a $\text{Spin}^c(4)$ -connection \mathbb{A} on P . Let us consider the pull-back $\text{Spin}^c(4)$ -connection $\pi^* \mathbb{A}$ on $\pi^* P \rightarrow \tilde{X}$. Since $\pi^* P = P_0 \cup P_1$ (see §3(iii)), the $\text{Spin}^c(4)$ -connection $\pi^* \mathbb{A}$ has a $\text{Spin}^c(4)$ -reduction $\tilde{\mathbb{A}}$ on the $\text{Spin}^c(4)$ -bundle P_0 . Then we obtain a $\text{U}(1)$ -connection \tilde{A} on L from $\tilde{\mathbb{A}}$, and we can see that

$$\mathcal{A}(E) \cong \mathcal{A}(L)^I.$$

The gauge transformation group on \tilde{X} is given by $\tilde{\mathcal{G}} = \text{Map}(\tilde{X}, S^1)$. If we define the involution I on $\tilde{\mathcal{G}}$ by $I\tilde{u} = \overline{\iota^* \tilde{u}}$ for $\tilde{u} \in \tilde{\mathcal{G}}$, then the \mathcal{G} -action on $\mathcal{A}(L) \times \Gamma(\tilde{S}^+)$ is I -equivariant, and $\tilde{\mathcal{G}}^I \cong \Gamma(\tilde{X} \times_{\{\pm 1\}} \text{U}(1)) \cong \mathcal{G}_0$. Let $\mathcal{C} = \mathcal{A}(E) \times \Gamma(S^+)$ and $\tilde{\mathcal{C}} = \mathcal{A}(L) \times \Gamma(\tilde{S}^+)$. Then we have

Proposition 4.10. $\mathcal{C}/\mathcal{G}_0 \cong \tilde{\mathcal{C}}^I/\tilde{\mathcal{G}}^I$.

Via the identifications above, the Spin^c -Dirac operator $D_A: \Gamma(S^+) \rightarrow \Gamma(S^-)$ can be identified with the restriction of the Spin^c -Dirac operator $D_{\tilde{A}}$ on (\tilde{X}, \tilde{c}) to the I -invariant part, $D_{\tilde{A}}: \Gamma(\tilde{S}^+)^I \rightarrow \Gamma(\tilde{S}^-)^I$. Furthermore, the Seiberg-Witten equations on (\tilde{X}, \tilde{c}) is I -equivariant in our setting. Let us define the I -invariant Seiberg-Witten moduli space $\tilde{\mathcal{M}}^I$ as the space of I -invariant solutions divided by $\tilde{\mathcal{G}}^I$. Then, it can be checked that $\tilde{\mathcal{M}}^I$ can be identified with the $\text{Pin}^-(2)$ -monopole moduli space \mathcal{M}_0 :

Proposition 4.11. $\mathcal{M}_0 \cong \tilde{\mathcal{M}}^I$.

Remark 4.12. The I -invariant moduli $\tilde{\mathcal{M}}^I$ can be embedded in the ordinary Seiberg-Witten moduli space $\tilde{\mathcal{M}}$ of (\tilde{X}, \tilde{c}) , since $\tilde{\mathcal{C}}^I/\tilde{\mathcal{G}}^I$ is continuously embedded in $\tilde{\mathcal{C}}/\tilde{\mathcal{G}}$ (cf. Remark 3.4 of [20] or [12]). Since $\tilde{\mathcal{M}}$ is compact, the compactness of \mathcal{M}_0 (Proposition 4.8) follows from Proposition 4.11, too.

Remark 4.13. In general, \mathcal{M} ($\tilde{\mathcal{M}}^I$) could be non-orientable. A similar but slightly different situation is studied by Tian-Wang [23]. They investigate the Seiberg-Witten theory in the presence of real structures on almost complex 4-manifolds. They also introduce an antilinear involution on the Seiberg-Witten theory. Their involution is different from ours in that they use the real structure to define the involution.

Remark 4.14. Since the Spin^c-Dirac operator D_A is the I -invariant part of $D_{\tilde{A}}$, the unique continuation theorem holds also for D_A . Of course, this can be proved directly.

4(vi). **The deformation complex.** When (A, Φ) is a solution of Pin⁻(2)-monopole equation, the deformation complex for \mathcal{M} (\mathcal{M}_0) at (A, Φ) is given as follows:

$$(4.15) \quad 0 \rightarrow \Omega^0(X; i\lambda) \xrightarrow{\alpha} \Omega^1(X; i\lambda) \oplus \Gamma(S^+) \xrightarrow{\beta} \Omega^+(X; i\lambda) \oplus \Gamma(S^-) \rightarrow 0,$$

where the maps α and β are the linearizations of the \mathcal{G} -action and the Pin⁻(2)-monopole equations, and given by $\alpha(f) = (-2df, f\Phi)$, $\beta(a, \phi) = (D_A\phi + \frac{1}{2}\rho(a)\Phi, d^+a - Dq_\Phi(\phi))$, where Dq_Φ is the linearization of q at Φ .

Let $(\tilde{A}, \tilde{\Phi})$ be the I -invariant solution on (\tilde{X}, \tilde{c}) corresponding to (A, Φ) . Then the deformation complex (4.15) can be identified with the restriction of the ordinary Seiberg-Witten deformation complex at $(\tilde{A}, \tilde{\Phi})$ to its I -invariant part:

$$(4.16) \quad 0 \rightarrow \Omega^0(\tilde{X}; i\mathbb{R})^I \rightarrow (\Omega^1(\tilde{X}; i\mathbb{R}) \oplus \Gamma(\tilde{S}^+))^I \rightarrow (\Omega^+(\tilde{X}; i\mathbb{R}) \oplus \Gamma(\tilde{S}^-))^I \rightarrow 0,$$

where the I -action on forms is given by the composition of the pullback by ι and the complex conjugation. For calculation of the index of (4.15), 0-th order terms can be neglected, and therefore, the complex (4.15) can be assumed to be a direct sum of the de Rham part and the Dirac part. (Cf. [19], 4.6.) The de Rham part is:

$$0 \rightarrow \Omega^0(X; i\lambda) \xrightarrow{d} \Omega^1(X; i\lambda) \xrightarrow{d^+} \Omega^+(X; i\lambda) \rightarrow 0.$$

The index of the Dirac part is calculated by applying the Lefschetz formula to the I -equivariant Dirac operator $D_{\tilde{A}}$ on (\tilde{X}, \tilde{c}) . More precisely, since the I -action is not complex linear, complexify the operator first, and then apply the Lefschetz formula [2]. Then the index of the Dirac part above is half of the index of $D_{\tilde{A}}$ because the ι -action on \tilde{X} is free. Thus we have,

Proposition 4.17. *The virtual dimension d of \mathcal{M} is given by*

$$(4.18) \quad d = \frac{1}{4}(\tilde{c}_1(E)^2 - \text{sign}(X)) - (b_0(X; l) - b_1(X; l) + b_+(X; l)),$$

where $\tilde{c}_1(E) \in H^2(X; l)$ is the twisted first Chern class. (See Remark 3.5.)

Remark 4.19. Note that $b_0(X; l) = 0$ if X is connected and l is nontrivial.

4(vii). **The topology of $\mathcal{C}^*/\mathcal{G}_0$.** Let \mathcal{C}^* be the space of irreducibles, i.e., $\mathcal{C}^* = \mathcal{A}(E) \times (\Gamma(\tilde{S}^+) \setminus 0)$. The purpose of this subsection is to prove the following proposition.

Proposition 4.20. *The space $\mathcal{C}^*/\mathcal{G}_0$ has the same homotopy type with*

$$\mathbb{RP}^\infty \times T^{b_1(X;\lambda)}.$$

The proof is divided into several steps.

Lemma 4.21. *The space \mathcal{C}^* is contractible.*

Proof. Note that $\mathcal{C}^* = \mathcal{A}(E) \times (\Gamma(\tilde{S}^+) \setminus 0)$ is the complement of a linear subspace with infinite codimension. Therefore \mathcal{C}^* has the homotopy type of an infinite dimensional sphere, and is contractible. \square

Since \mathcal{G}_0 acts on \mathcal{C}^* freely, Lemma 4.21 implies that $\mathcal{C}^*/\mathcal{G}_0$ has the homotopy type of the classifying space $B\mathcal{G}_0$. Hence, Proposition 4.20 follows from the next lemma.

Lemma 4.22. $\mathcal{G}_0 \simeq (\mathbb{Z}/2) \times \mathbb{Z}^{b_1(X,l)}$.

Proof. We will prove $\pi_0\mathcal{G}_0$ is isomorphic to $H^1(X;l)$. Then the lemma follows because $H^1(X;l) = \mathbb{Z}/2 \oplus \mathbb{Z}^{b_1(X,l)}$ which is proved by the universal coefficient theorem. To prove the isomorphism $\pi_0\mathcal{G}_0 \cong H^1(X;l)$, one can use obstruction theory. As an alternative way of the proof, we use sheaf cohomology. Let us define the bundles λ and κ over X by $\lambda = l \otimes \mathbb{R}$ and $\kappa = \tilde{X} \times_{\{\pm 1\}} S^1$, and let $\mathcal{C}^\infty(\lambda)$ and $\mathcal{C}^\infty(\kappa)$ be the sheaves on X of germs of C^∞ -sections of λ and κ , respectively. Then there is the short exact sequence of sheaves:

$$1 \rightarrow l \rightarrow \mathcal{C}^\infty(\lambda) \rightarrow \mathcal{C}^\infty(\kappa) \rightarrow 1.$$

The long exact sequence is induced:

$$\begin{aligned} 0 \rightarrow H^0(X;l) \rightarrow H^0(X;\mathcal{C}^\infty(\lambda)) \rightarrow H^0(X;\mathcal{C}^\infty(\kappa)) \\ \rightarrow H^1(X;l) \rightarrow H^1(X;\mathcal{C}^\infty(\lambda)) \rightarrow H^1(X;\mathcal{C}^\infty(\kappa)) \rightarrow \cdots \end{aligned}$$

Now, the lemma follows because $H^i(X;\mathcal{C}^\infty(\lambda)) = 0$ and $\pi_0\mathcal{G}_0 \cong H^0(X;\mathcal{C}^\infty(\kappa))$. \square

As mentioned above, the \mathcal{G}_0 -action on $\mathcal{A}(E)$ is not free. We will need a subgroup of \mathcal{G}_0 which acts on $\mathcal{A}(E)$ freely defined as follows. Let us take a closed loop $\gamma: S^1 \rightarrow X$ so that the restriction of λ to γ , $\lambda|_\gamma = \gamma^*\lambda$, is a nontrivial \mathbb{R} -bundle over γ . Let $\tilde{\gamma} \rightarrow \gamma$ be the connected double covering of γ . Let us define \mathcal{G}_γ by $\mathcal{G}_\gamma = \Gamma(\tilde{\gamma} \times_{\{\pm 1\}} \mathrm{U}(1))$ where the $\{\pm 1\}$ -action on $\mathrm{U}(1)$ is given by complex conjugation. Then \mathcal{G}_γ has the following properties:

- By restricting \mathcal{G}_0 to γ , we have a surjective homomorphism $\mathcal{G}_0 \rightarrow \mathcal{G}_\gamma$.
- \mathcal{G}_γ has two components. Therefore $\pi_0\mathcal{G}_\gamma \cong \{\pm 1\}$.
- By restriction and projection, we have a surjective homomorphism

$$\theta_\gamma: \mathcal{G}_0 \rightarrow \pi_0\mathcal{G}_\gamma = \{\pm 1\}.$$

Let us define $\mathcal{K}_\gamma = \ker \theta_\gamma$.

Remark 4.23. Let $\{\pm 1\}$ be the subgroup of constant sections in \mathcal{G}_0 , and let us consider the exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G}_0/\{\pm 1\} \rightarrow 1.$$

Then $\mathcal{G}_0/\{\pm 1\}$ is homotopy equivalent to $\mathbb{Z}^{b_1(X;l)}$, and the map θ_γ gives a splitting of the sequence.

4(viii). **The cut-down moduli space.** Since the moduli space \mathcal{M}_0 is not necessarily a manifold, we need to perturb the equations. As in the Seiberg-Witten case, we will perturb the second equation of (4.4) by adding an $i\lambda$ -valued self-dual 2-form. On the other hand, as we will see later (§6(i)), the whole theory of Pin⁻(2)-monopole equations can be considered as a family over a torus $T^{b_1(X;\lambda)}$. For our purpose, we will cut down the moduli space along a fiber over a point in $T^{b_1(X;\lambda)}$. These are the tasks of this subsection.

Let us define the \mathcal{G}_0 -equivariant map

$$\hat{\mu}: L_k^2(\mathcal{A}(E) \times \Gamma(S^+)) \times L_{k-1}^2(\Omega^+(i\lambda)) \rightarrow L_{k-1}^2(\Gamma(S^-) \times \Omega^+(i\lambda))$$

by

$$\hat{\mu}(A, \Phi, \eta) = (D_A \Phi, F_A^+ - q(\Phi) - \eta),$$

where \mathcal{G}_0 acts on $\Omega^+(i\lambda)$ trivially, and \mathcal{G}_0 is completed by L_{k+1}^2 . We suppose $k \geq 4$ so that $L_{k-1}^2 \subset C^0$. (Below we omit the symbol L_m^2 .) Let us fix a reference connection $A_0 \in \mathcal{A}(E)$. Then $\mathcal{A}(E)$ can be identified with $A_0 + \Omega^1(i\lambda)$. Let us consider the L^2 -orthogonal splitting: $\Omega^1(i\lambda) = \ker d \oplus (\ker d)^\perp$. Then $\hat{\mu}$ can be considered as a map from $\ker d \times (\ker d)^\perp \times \Gamma(S^+) \times \Omega^+(i\lambda)$.

Proposition 4.24. *Suppose $b^+(X; \lambda) = 0$. If $\text{ind } D_{A_0} \geq 0$, then there exists a gauge invariant open-dense subset \mathcal{U} of $\ker d \times \Omega^+(i\lambda)$ which has the property that the restriction of $\hat{\mu}$ to $\mathcal{U}' = \mathcal{U} \times (\ker d)^\perp \times \Gamma(S^+)$ has 0 as regular value.*

To prove Proposition 4.24, we use the next lemma.

Lemma 4.25. *Suppose $\text{ind } D_{A_0} \geq 0$, and let \mathcal{O} be the set of $A \in \mathcal{A}(E)$ such that D_A is surjective. Then \mathcal{O} is a gauge invariant open-dense subset of $\mathcal{A}(E)$.*

Although the proof of this lemma is standard, we will give a proof for reader's convenience. The proof is divided into several steps. (Cf. [19], Chapter 6 and [18], §3.4.)

Let us define $F: \mathcal{A}(E) \times \Gamma(S^+) \rightarrow \Gamma(S^-)$ by $F(A, \Phi) = D_A \Phi$. Then the differential of F is given by

$$DF_{(A, \Phi)}(a, \phi) = D_A \phi + \frac{1}{2} \rho(a) \Phi, \quad \text{for } (a, \phi) \in \Omega^1(i\lambda) \times \Gamma(S^+).$$

Lemma 4.26. *If $F(A, \Phi) = 0$ and $\Phi \neq 0$, then $DF_{(A, \Phi)}$ is surjective.*

Proof. First, note that, if $\phi_x \in S_x^+$, a spinor vector over $x \in X$, is nonzero, the linear map

$$T_x^* X \ni a_x \mapsto \rho(a_x) \phi_x \in S_x^+$$

is an isomorphism. Suppose $\psi \in \Gamma(S^-)$ is perpendicular to the image of $DF_{(A, \Phi)}$. Then by holding $\phi = 0$ and varying a , we see $\psi = 0$ on the support U of Φ which is open. By

holding $a = 0$ and varying ϕ , we have $D_A\psi = 0$, and by the unique continuation theorem, ψ is identically zero. Thus the lemma is proved. \square

By the implicit function theorem, $N = F^{-1}(0) \cap \{\Phi \neq 0\}$ is a submanifold of $\mathcal{A}(E) \times \Gamma(S^+)$. Let $\pi_0: N \rightarrow \mathcal{A}(E)$ be the projection to the first factor, $\pi_0(A, \Phi) = A$. By the standard argument, it is easy to see the following. (See e.g. [21], §1.5.2.)

Lemma 4.27. *The map π_0 is Fredholm whose index is equal to $\text{ind } D_{A_0}$.*

Now let us prove Lemma 4.25.

Proof of Lemma 4.25. Suppose $\text{ind } D_{A_0}$ is nonnegative and let \mathcal{O} be the set of regular values of π_0 . Then $A \in \mathcal{O}$ if and only if D_A is surjective. By the Sard-Smale theorem, \mathcal{O} is a dense subspace of $\mathcal{A}(E)$.

The space \mathcal{O} is gauge invariant because F and π_0 is gauge equivariant.

Let us prove \mathcal{O} is open. Note that D_A is surjective if and only if its formal adjoint $D_A^*: L_k^2(S^-) \rightarrow L_{k-1}^2(S^+)$ is injective. That is,

$$\mathcal{O} = \{A \in \mathcal{A}(E) \mid \ker D_A^* = 0\}.$$

Let $\bar{\mathcal{O}}$ be the complement of \mathcal{O} in $\mathcal{A}(E)$. Let us prove $\bar{\mathcal{O}}$ is closed. Recall $\mathcal{A}(E)$ is topologized by L_k^2 for fixed $k \geq 4$. Suppose it is given a sequence $\{A_i\}_{i=1,2,\dots} \subset \bar{\mathcal{O}}$ such that

- $A_i \rightarrow A$ for some $A \in \mathcal{A}(E)$ in L_k^2 , and
- there exists a sequence $\{\phi_i\} \subset \Gamma(S^-)$ which satisfies $D_{A_i}^*\phi_i = 0$ and $\|\phi_i\|_{L_k^2} = 1$.

Let $a_i = A_i - A$. By the elliptic regularity, there exists some constant C such that

$$\|\phi_i\|_{L_{k+1}^2} \leq C(\|\phi_i\|_{L_k^2} + \|D_{A_i}^*\phi_i\|_{L_k^2}).$$

Since $D_{A_i}^*\phi_i = -\frac{1}{2}\rho(a_i)\phi_i$ and a_i and ϕ_i is L_k^2 -bounded, $D_{A_i}^*\phi_i$ is also L_k^2 -bounded. Therefore ϕ_i is L_{k+1}^2 -bounded. By Rellich's theorem, a subsequence of $\{\phi_i\}$ converge to some ϕ in L_k^2 with norm 1. \square

Let us prove Proposition 4.24.

Proof of Proposition 4.24. (Cf. [18], §3.4.) The differential of $\hat{\mu}$,

$$D\hat{\mu}_{(A,\Phi,\eta)}: \ker d \times (\ker d)^\perp \oplus \Gamma(S^+) \oplus \Omega^+(i\lambda) \rightarrow \Gamma(S^-) \oplus \Omega^+(i\lambda)$$

is given by

$$D\hat{\mu}_{(A,\Phi,\eta)}(a, b, \phi, \sigma) = (D_A\phi + \frac{1}{2}\rho(a+b)\Phi, d^+b - Dq_\Phi(\phi) - \sigma).$$

Let us consider the subset $\mathcal{U} \subset \ker d \times \Omega^+(i\lambda)$ consisting of $(a, \eta) \in \ker d \times \Omega^+(i\lambda)$ satisfying the following property:

If $b \in (\ker d)^\perp$ satisfies $F_{A_0}^+ + d^+b = \eta$, then D_{A_0+a+b} is surjective.

Since the restriction of d^+ to $(\ker d)^\perp$ is a linear homeomorphism between $(\ker d)^\perp$ and $\Omega^+(i\lambda)$ if $b_+(X; \lambda) = 0$, the space \mathcal{U} is gauge invariant and open-dense by Lemma 4.25. Now we claim that, if (A, Φ, η) is a solution to the equation $\hat{\mu} = 0$, the differential $D\hat{\mu}_{(A,\Phi,\eta)}|_{\mathcal{U}}$

is surjective: We may assume $0 \in \mathcal{U}'$. Suppose $(\psi, c) \in \Gamma(S^+) \times \Omega^+(i\lambda)$ is perpendicular to the image of $D\hat{\mu}_{(A, \Phi, \eta)}|_{\mathcal{U}'}$. By holding $\phi = 0$ and $a = b = 0$ and varying σ , we obtain c must be 0. If $\Phi \not\equiv 0$, then it can be proved that $\psi \equiv 0$ by the unique continuation theorem as in the proof of Lemma 4.26. If $\Phi \equiv 0$, then ψ must be 0 by the definition of \mathcal{U}' . \square

By Proposition 4.24 and the implicit function theorem,

$$\mathcal{Z} = \{(A, \Phi, \eta) \in \mathcal{U}' \mid \hat{\mu}(A, \Phi, \eta) = 0\}$$

is a submanifold in \mathcal{U}' .

Let us consider the projection:

$$\tilde{\pi}: \ker d \times (\ker d)^\perp \times \Gamma(S^+) \times \Omega^+(i\lambda) \rightarrow \ker d \times \Omega^+(i\lambda).$$

Let us take a subgroup \mathcal{K}_γ as in §4(vii). Note that \mathcal{K}_γ acts freely on $\ker d$, and $\ker d/\mathcal{K}_\gamma$ is isomorphic to a $b^1(X; \lambda)$ -dimensional torus $T^{b_1(X; \lambda)}$. Recall that \mathcal{U}' is gauge invariant. Then, by restricting $\tilde{\pi}$ to \mathcal{Z} and dividing it by \mathcal{K}_γ , we obtain a map,

$$\pi: \mathcal{Z}/\mathcal{K}_\gamma \rightarrow T^{b_1(X; \lambda)} \times \Omega^+(i\lambda).$$

This is a smooth map between Banach manifolds. As in the Seiberg-Witten case, we can prove the following:

Proposition 4.28. *The map π is a Fredholm map whose index is*

$$d' = d - b^1(X; \lambda) = \text{ind } D_{A_0} = \frac{1}{4}(\tilde{c}_1(E)^2 - \text{sign}(X)).$$

Proof. The local slice of \mathcal{K}_γ -action at (A, Φ) is given by the set of elements

$$(\alpha, \phi, \sigma) \in \Omega^1(i\lambda) \times \Gamma(S^+) \times \Omega^+(i\lambda)$$

which are L^2 -perpendicular to $(-2du, u\Phi, 0)$ for every $u \in \Omega^0(i\lambda)$. Let us define $f(\phi, \Phi) \in \Omega^0(i\lambda)$ by the relation

$$\langle \phi, u\Phi \rangle_{\mathbb{R}} = \langle f(\phi, \Phi), u \rangle_{i\lambda},$$

where $\langle \cdot, \cdot \rangle_{i\lambda}$ is the natural metric on $i\lambda = i(l \otimes \mathbb{R})$. The tangent space of $\mathcal{Z}/\mathcal{K}_\gamma$ is identified with the kernel of the map

$$F: \Omega^1(i\lambda) \times \Gamma(S^+) \times \Omega^+(i\lambda) \rightarrow \Omega^0(i\lambda) \times \Gamma(S^-) \times \Omega^+(\lambda)$$

defined by

$$F(\alpha, \phi, \eta) = (-2d^*\alpha + f(\phi, \Phi), D\hat{\mu}(\alpha, \phi, \sigma)).$$

Then, it follows from the standard argument (e.g. [21], §1.5.2) that π is Fredholm, and the index of π is given by the sum of the index of D_A and the index of the restriction of $d^* + d^+$ to the L^2 -complement of the space of harmonic 1-forms. \square

By the Sard-Smale theorem, for a generic choice of $(t, \eta) \in T^{b_1(X; \lambda)} \times \Omega^+(i\lambda)$, we obtain a d' -dimensional manifold

$$\mathcal{M}'(t, \eta) = \pi^{-1}(t, \eta) \subset \mathcal{C}/\mathcal{K}_\gamma.$$

The quotient group $\mathcal{G}_0/\mathcal{K}_\gamma \cong \{\pm 1\}$ still acts on $\mathcal{M}'(t, \eta)$, and there exists a unique fixed point. Then the quotient space $\mathcal{M}(t, \eta) = \mathcal{M}'(t, \eta)/\{\pm 1\}$ is a V -manifold which has a

unique quotient singularity. Around the singularity, we can take an open neighborhood N of the form of a cone of $\mathbb{RP}^{d'-1}$. Removing N from $\mathcal{M}(t, \eta)$, we obtain a d' -dimensional compact manifold

$$\overline{\mathcal{M}}(t, \eta) = \mathcal{M}(t, \eta) \setminus N,$$

whose boundary is $\mathbb{RP}^{d'-1}$.

Now, we prove the lemma which will be a key point of our argument.

Lemma 4.29. *If $b_+(X; l) = 0$, then $d' \leq 0$.*

Proof. Suppose $d' = \text{ind } D_{A_0} > 0$. We obtain a d' -dimensional compact manifold $\overline{\mathcal{M}}(t, \eta)$ as above. Note that $\overline{\mathcal{M}}(t, \eta) \subset \mathcal{C}^*/\mathcal{G}_0$ and $\partial(\overline{\mathcal{M}}(t, \eta)) \cong \mathbb{RP}^{d'-1}$. Then there exists a class $A \in H^{d'-1}(\mathcal{C}^*/\mathcal{G}_0; \mathbb{Z}/2) = H^{d'-1}(\mathbb{RP}^\infty \times T^{b_1(X; \lambda)}; \mathbb{Z}/2)$ so that $\langle A, [\partial(\overline{\mathcal{M}}(t, \eta))] \rangle \neq 0$. This is a contradiction. \square

5. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Suppose that X and l satisfy the conditions in Theorem 1.1.

Definition 5.1. An element w in a lattice L is called characteristic if $w \cdot v \equiv v \cdot v \pmod{2}$ for any $v \in L$.

Lemma 5.2. *The second Stiefel-Whitney class $w_2(X)$ has a lift in $H^2(X; l)$. Moreover, for every class $c \in H^2(X; l)$ whose class $[c]$ in $H^2(X; l)/\text{torsion}$ is a characteristic element of $Q_{X, l}$, there exists a torsion class $\delta \in H^2(X; l)$ such that $c + \delta$ is a lift of $w_2(X)$.*

Proof. (Cf. [1].) Note that $l^* = l$ and $\text{Hom}(l; \mathbb{Z}/2) = \mathbb{Z}/2$. By the universal coefficient theorem, we have a commutative diagram,

$$\begin{array}{ccccc} \text{Ext}(H_1(X; l), \mathbb{Z}) & \longrightarrow & H^2(X; l) & \xrightarrow{h_1} & \text{Hom}(H_2(X; l), \mathbb{Z}) \\ \downarrow \rho_0 & & \downarrow \rho_1 & & \downarrow \rho_2 \\ \text{Ext}(H_1(X; l), \mathbb{Z}/2) & \xrightarrow{k} & H^2(X; \mathbb{Z}/2) & \xrightarrow{h_2} & \text{Hom}(H_2(X; l), \mathbb{Z}/2). \end{array}$$

Note that the homomorphisms h_1 and h_2 are given as follows: Let $[X] \in H^4(X; \mathbb{Z})$ and $[X]_2 \in H^4(X; \mathbb{Z}/2)$ be the fundamental classes in coefficients \mathbb{Z} and $\mathbb{Z}/2$. For $a \in H^2(X; l)$, $a' \in H^2(X; \mathbb{Z}/2)$ and $\beta \in H_2(X; l)$,

$$h_1(a)(\beta) = \langle a \cup b, [X] \rangle, \quad h_2(a')(\beta) = \langle a' \cup b, [X]_2 \rangle,$$

where $b \in H^2(X; l)$ is the Poincaré dual of β , $a' \cup b$ is defined by the cup product

$$H^2(X; \mathbb{Z}/2) \otimes H^2(X; l) \rightarrow H^2(X; l \otimes \mathbb{Z}/2) = H^2(X; \mathbb{Z}/2).$$

Let $S \in \text{Hom}(H_2(X; l), \mathbb{Z}/2)$ be the homomorphism defined by

$$S(\beta) = \langle b \cup b, [X] \rangle \pmod{2},$$

where $\beta \in H_2(X; l)$ and $b \in H^2(X; l)$ is the Poincaré dual of β . If $c \in H^2(X; l)$ satisfies the assumption of the lemma, then $h_2\rho_1(c) = \rho_2h_1(c) = S(\gamma)$, where γ is the Poincaré dual of

c. On the other hand, $h_2(w_2(X)) = S$ by Wu's formula. Therefore $h_2(w_2(X) - \rho_1(c)) = 0$, and there exists a class $\delta' \in \text{Ext}(H_1(X; l); \mathbb{Z}/2)$ such that

$$w_2(X) - \rho_1(c) = k(\delta').$$

Since ρ_0 is surjective, there exists a lift $\delta \in \text{Ext}(H_1(X; l); \mathbb{Z})$ such that $\rho_0(\delta) = \delta'$, and this is a required δ . \square

Theorem 5.3. *Let X be a closed, connected, oriented smooth 4-manifold. Suppose we have a nontrivial \mathbb{Z} -bundle $l \rightarrow X$ satisfying $b_+(X; l) = 0$. Let $\lambda = l \otimes \mathbb{R}$. Then, for every cohomology class $C \in H^2(X; l)$ which satisfies $[C]_2 + w_1(\lambda)^2 = w_2(X)$, where $[C]_2$ is the mod 2 reduction of C , the inequality $|C^2| \geq b_2(X; l)$ holds.*

Proof. If $b_+(X; l) = 0$, then $C^2 \leq 0$ for $C \in H^2(X; l)$ and $\text{sign}(X) = -b_2(X; l)$. For $C \in H^2(X; l)$ satisfying the assumption, there is a Spin^{c-} -structure on X whose $\text{O}(2)$ -bundle E has $\tilde{c}_1(E) = C$ by Proposition 3.4. Let us consider the $\text{Pin}^-(2)$ -monopole moduli space on the Spin^{c-} -structure. Then Lemma 4.29 implies that $d' = 1/4(C^2 - \text{sign}(X)) = 1/4(C^2 + b_2(X; l)) \leq 0$. Thus, $|C^2| \geq b_2(X; l)$ holds. \square

To complete the proof of Theorem 1.1, we invoke the following theorem due to Elkies.

Theorem 5.4 (Elkies[9]). *Let L be a definite unimodular form over \mathbb{Z} . If every characteristic element $w \in L$ satisfies $|w^2| \geq \text{rank } L = n$, then L is isomorphic to the standard form \mathbb{Z}^n .*

Proof of Theorem 1.1. We can assume that $b_+(X; l) = 0$ by reversing the orientation if necessary. Under the assumptions of Theorem 1.1, Wu's formula, Lemma 5.2 and Theorem 5.3 imply that every characteristic element C of $Q_{X, l}$ satisfies $|C^2| \geq \text{rank } Q_{X, l}$. Then, by Elkies' theorem, $Q_{X, l}$ should be the standard form. \square

6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using the technique of the finite dimensional approximation [3] and equivariant K -theory as in Bryan's paper [5]. We also give an alternative proof of Theorem 1.1 by the same technique.

6(i). **The $\text{Pin}^-(2)$ -monopole map.** Let us introduce the $\text{Pin}^-(2)$ -monopole map $\tilde{\mu}$ defined as follows (Cf. [3], p.11):

$$\begin{aligned} \tilde{\mu}: \mathcal{A}(E) \times (\Gamma(S^+) \oplus \Omega^1(X; i\lambda)) \\ \rightarrow \mathcal{A}(E) \times (\Gamma(S^-) \oplus \Omega^+(X; i\lambda) \oplus \Omega^0(X; i\lambda) \oplus H^1(X; i\lambda), \\ (A, \phi, a) \mapsto (A, D_{A+a}\phi, F_A^+ + d^+a - q(\phi), d^*a, a_{\text{harm}}), \end{aligned}$$

where a_{harm} is the harmonic part of a . When $\tilde{c}_1(E) \neq 0$, let $\mathcal{G} = \mathcal{G}_0$ act trivially on forms. When $\tilde{c}_1(E) = 0$, let \mathcal{G} act on forms by multiplication of ± 1 via the projection $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_0 \cong \{\pm 1\}$. Then the monopole map $\tilde{\mu}$ is \mathcal{G} -equivariant.

Let us choose a reference connection A and take a subgroup $\mathcal{K}_\gamma \subset \mathcal{G}_0$ as in §4(vii). The subspace $A + \ker d \subset \mathcal{A}(E)$ is preserved by the action of \mathcal{K}_γ , and the \mathcal{K}_γ -action is free.

The quotient space is isomorphic to the torus $T^{b_1(X;l)} = H^1(X; \lambda)/H^1(X; l)$. Let \mathcal{V} and \mathcal{W} be the quotient spaces,

$$\begin{aligned}\mathcal{V} &= (A + \ker d) \times (\Gamma(S^+) \oplus \Omega^1(X; i\lambda))/\mathcal{K}_\gamma, \\ \mathcal{W} &= (A + \ker d) \times (\Gamma(S^-) \oplus \Omega^+(X; i\lambda) \oplus \Omega^0(X; i\lambda) \oplus H^1(X; i\lambda))/\mathcal{K}_\gamma.\end{aligned}$$

Then \mathcal{V} and \mathcal{W} are bundles over $T^{b_1(X;l)}$. Dividing $\tilde{\mu}$ by \mathcal{K}_γ , we obtain a fiber preserving map

$$\mu = \tilde{\mu}/\mathcal{K}_\gamma: \mathcal{V} \rightarrow \mathcal{W}.$$

Then $\mathcal{G}_0/\mathcal{K}_\gamma = \{\pm 1\}$ still acts on \mathcal{V} and \mathcal{W} , and μ is a $\mathbb{Z}/2$ -equivariant map in general. If $\tilde{\mu}_1(E) = 0$, take a flat connection on $E \cong \underline{\mathbb{R}} \oplus \lambda$ as a reference connection which is the product connection of flat connections on $\underline{\mathbb{R}}$ and λ . Then μ is a $\mathbb{Z}/4$ -equivariant map.

For a fixed $k > 4$, we take the fiberwise L_k^2 -completion of \mathcal{V} and the fiberwise L_{k-1}^2 -completion of \mathcal{W} . Then we can prove the map μ is a Fredholm proper map as in [3]. In fact, we can readily prove the following by using the Weitzenböck formula (4.9).

Proposition 6.1 ([3]). *Preimages $\mu^{-1}(B) \subset \mathcal{V}$ of bounded disk bundles $B \subset \mathcal{W}$ are contained in bounded disk bundles.*

With this understood, we can construct a finite dimensional approximation $f: V \rightarrow W$ of μ between some finite rank vector bundles over $T^{b_1(X;l)}$ as in [3]. The map f is also a $\mathbb{Z}/2$ (or $\mathbb{Z}/4$)-equivariant proper map.

Remark 6.2. As mentioned in Remark 1.3(3), the $\text{Pin}^-(2)$ -monopole map can be identified with the I -invariant part of the ordinary Seiberg-Witten monopole map. To see this, we need a little care on the gauge transformation group because the based gauge group which is used in the Seiberg-Witten monopole map is not compatible to \mathcal{K}_γ . However, by constructing another subgroup compatible to \mathcal{K}_γ , we can obtain such an identification. This issue will be discussed elsewhere.

Remark 6.3. We can further develop $\text{Pin}^-(2)$ -monopole gauge theory. Many things in the Seiberg-Witten theory could also be considered in the $\text{Pin}^-(2)$ -monopole theory. Especially, we can define $\text{Pin}^-(2)$ -monopole invariants and their cohomotopy refinements. It would be also interesting to consider gluing formulas, Floer theory, and so on. All of these issues are left to future researches.

6(ii). **Equivariant K -theory.** We review several facts on equivariant K -theory, especially, the equivariant Thom isomorphism and tom Dieck's character formula for the K -theoretic degree. We refer to the readers §3.3 of [5] and tom Dieck's book [6], pp.254–255.

Let V and W be complex Γ -representations for some compact Lie group Γ . Let BV and BW be Γ -invariant balls in V and W and let $f: BV \rightarrow BW$ be a Γ -map preserving the boundaries SV and SW . The K -group $K_\Gamma(V)$ is defined as $K_\Gamma(BV, SV)$, and the equivariant Thom isomorphism theorem says that $K_\Gamma(V)$ is a free $R(\Gamma)$ -module with the Bott class $\lambda(V)$ as generator, where $R(\Gamma)$ is the complex representation ring of Γ . The map f induces a homomorphism $f^*: K_\Gamma(W) \rightarrow K_\Gamma(V)$. The K -theoretic degree $\alpha_f \in R(\Gamma)$ is uniquely determined by the relation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$.

For $g \in \Gamma$, let V_g and W_g be the subspaces of V and W fixed by g , and let V_g^\perp and W_g^\perp be their orthogonal complements. Let $f^g: V_g \rightarrow W_g$ be the restriction of f , and let $d(f^g)$ be the ordinary topological degree of f^g . (Note that $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$.) For $\beta \in R(\Gamma)$, let $\Lambda_{-1}\beta$ be the alternating sum $\sum (-1)^i \Lambda^i \beta$ of exterior powers.

Then tom Dieck's character formula [6] is,

$$(6.4) \quad \text{tr}_g(\alpha_f) = d(f^g) \text{tr}_g(\Lambda_{-1}(W_g^\perp - V_g^\perp)),$$

where tr_g is the trace of the g -action.

6(iii). **Proof of Theorem 1.2.** Suppose X and a \mathbb{Z} -bundle l satisfy the assumptions of Theorem 1.2. Let $\lambda = l \otimes \mathbb{R}$ and $E = \underline{\mathbb{R}} \oplus \lambda$. By the assumptions, a Spin^c -structure (P, τ) for (X, E) exists by Proposition 3.4, and we obtain a finite dimensional approximation $f: V \rightarrow W$ of the Pin⁻(2)-monopole map on (P, τ) . Since $\tilde{c}_1(E) = 0$, f is a $\Gamma = \mathbb{Z}/4$ -equivariant proper map. If $b_1(X; l) > 0$, by restricting f to the fiber over the origin of $T^{b_1(X; l)}$ which is represented by the fixed reference connection A , f can be assumed to be a Γ -map between (real) Γ -representation V and W . In fact, f can be considered as a map of the following form,

$$f: \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \rightarrow \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

where $\Gamma = \mathbb{Z}/4$ acts on $\tilde{\mathbb{R}}$ by multiplication of ± 1 via the surjection $\mathbb{Z}/4 \rightarrow \{\pm 1\}$, and on \mathbb{C}_k by multiplication of $g = \exp 2\pi\sqrt{-1}k/4$ for some fixed generator g of Γ , m, n are some positive integers, $b = b_+(X; l)$ and

$$k = \frac{1}{2} \text{ind}_{\mathbb{R}} D_A = \frac{1}{8} (\tilde{c}_1(E)^2 - \text{sign}(X)) = -\frac{1}{8} \text{sign}(X).$$

As in [13], take the complexification of f as $f(u \otimes 1 + v \otimes i) = f(u) \otimes 1 + f(v) \otimes i$. Now the complexified f is of the form,

$$f: \tilde{\mathbb{C}}^m \oplus (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^{n+k} \rightarrow \tilde{\mathbb{C}}^{m+b} \oplus (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^n,$$

where $\tilde{\mathbb{C}} = \tilde{\mathbb{R}} \otimes \mathbb{C}$. Let us apply tom Dieck's formula (6.4). Since $V_g = W_g = 0$, $d(f^g) = 1$. Then we have,

$$\text{tr}_g(\alpha_f) = \text{tr}_g(\Lambda_{-1}(\tilde{\mathbb{C}}^b - (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^k)) = \text{tr}_g((\mathbb{C} - \tilde{\mathbb{C}})^b (2\mathbb{C} - \mathbb{C}_1 \oplus \mathbb{C}_{-1})^{-k}) = 2^{b-k}.$$

Since $\text{tr}_g(\alpha_f)$ is an integer, we have $b - k \geq 0$. Thus, Theorem 1.2 is proved.

Remark 6.5. In the proof of Theorem 1.2, we restrict the finite dimensional approximation f to a fiber, and take the complexification of it. Due to such modifications of f , the inequality we obtained might be somewhat weaker than expected. One could improve the inequality by using the technique of [14].

6(iv). **An alternative proof of Theorem 1.1.** In this subsection, we give an alternative proof of Theorem 1.1 by giving an alternative proof of Lemma 4.29. Suppose X and l satisfy the assumption of Theorem 1.1. We may assume $b_+(X; l) = 0$ by reversing the orientation of X if necessary. Let E be an $\text{O}(2)$ -bundle such that $\det E = \lambda$, and suppose a Spin^c -structure on (X, E) is given. Then we have a $\Gamma = \mathbb{Z}/2$ -equivariant finite dimensional

approximation $f: V \rightarrow W$ of the $\text{Pin}^-(2)$ -monopole map. By restricting f to a fiber if $b_1(X; l) > 0$, we may assume f has the form of

$$f: \mathbb{R}^m \oplus \tilde{\mathbb{C}}^n \rightarrow \mathbb{R}^m \oplus \tilde{\mathbb{C}}^{n+k},$$

where $\Gamma \cong \{\pm 1\}$ acts on \mathbb{R} trivially, and on $\tilde{\mathbb{C}}$ by multiplication of ± 1 , and m, n are some positive integers, and

$$k = -\frac{1}{2} \text{ind}_{\mathbb{R}} D_A = -\frac{1}{8} (\tilde{c}_1(E)^2 - \text{sign}(X)).$$

Take the complexification of f and apply tom Dieck's formula (6.4) for $g = -1$. Then,

$$\text{tr}_g(\alpha_f) = \text{tr}_g((\mathbb{C} - \tilde{\mathbb{C}})^{2k}) = 2^{2k}.$$

Therefore $k \geq 0$, and Lemma 4.29 is proved.

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